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Decay rates of Saint-Venant type for functionally graded heat-conducting hollowed cylinder

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Abstract

In this paper we consider the case of functionally graded heat-conducting hollowed cylinder. Our purpose is to investigate the consequences of the material inhomogeneity on the decay of Saint-Venant end effects in the case of linear isotropic rigid solids. The mathematical issues involve the implications of spatial inhomogeneity on the decay rates of solutions to Dirichlet boundary-value problems. The rate of decay is characterized in terms of the smallest eigenvalue of a Sturm-Liouville problem. We first consider the case where the inhomogeneity depends on the radius of the cross-section, but later we also consider the case where the inhomogeneity also depends on the axial variable. Last section considers the case where the cross-section is increasing. Some tables and pictures illustrate our estimates.

keywords: Functionally graded materials, heat conduction, spatial decay estimates, Saint-Venant's principle, inhomogeneity

1 Introduction

The task to determine how the perturbations imposed on a part of the boundary of an elastic body are damped far away from the place where they were imposed is usually known as Saint-Venant's principle [1, 2]. This principle was primarily considered for static elastic materials, but currently it is applied for every kind of thermomechanical question that can be modelled by means of partial differential equations (see, for instance [3–9]). In particular, it is relevant to obtain the rate of decay of the perturbations in order to evaluate and clarify the influence far away where they were imposed. Particularly, this question is also considered for static and dynamic thermal problems (see [10]).

The influence of the material inhomogeneity is a topic which is deserving much attention in the last years. These materials are characterized by the continuous varying properties tailored to satisfy particular applications in engineering. It is worth noting that this kind of studies is motivated by the big interest on functionally graded materials to be applied to enginering.

In particular, we can cite several works where the influence of the material inhomogeneity of the rate of decay Saint-Venant type has been investigated in elasticity, mixtures, piezoelectric materials and heat conducting material [11–21].

In this paper we consider a heat-conducting functionally graded hollowed cylinder and we want to obtain rates of decay of the solutions for such materials. We first assume that the inhomogeneity depends on the radial coordinate of the cross section. We will see that, in this case, the rate of decay is obtained by means of the first eigenvalue of a Sturm-Liouville problem. We here adapt a mathematical result to our situation to obtain our aim. Later, we also consider the case when the inhomogeneity also depends on the axial coordinate and we adapt our studies to obtain a rate of decay. Examples in both situations represent an innovation aspect of our work.

The plan of this paper is the following. In the next section we define the problem we want to investigate in the case where the inhomogeneity only depends on the radial variable of the cross-section. We also obtain a first estimate for the decay which is characterized as the first eigenvalue of a Sturm-Liouville problem. In Section 3 we propose a theorem and two corollaries which give a lower bound for the first eigenvalue of the Sturm-Liouville problem. These results are applied to several cases in Section 4. Some tables and pictures illustrate the results. In Section 5 we also consider the case where the inhomogeneity also depends on the axial variable. We apply our results to three interesting examples. Section 6 is devoted to the case when the cross-section (circular crown) is an increasing function of the axial direction.

2 Basic Equations

In this section we consider the mathematical aspect of the heat conducting problem. First, we will work in the case of a semi-infinite hollowed cylinder $R = D \times [0, \infty)$, where D is a circular crown determined by the circles of radius a and b, where $0 < a < b$.

We first assume an inhomogeneity in the sense that the thermal conductivity depends on the radial variable of the cross section

$$r = \left(x_1^2 + x_2^2\right)^{1/2}. \tag{2.1}$$

We then study the rate of decay for the solutions of the problem determined by the equation

$$\left(K(r)u_{,i}\right)_{,i} = 0 \quad \text{in} \quad R, \tag{2.2}$$

with the boundary conditions

$$u(\boldsymbol{x}) = 0 \quad \text{on } \partial D \times [0, \infty), \tag{2.3}$$

$$u(x_1, x_2, 0) = f(x_1, x_2) \quad \text{on } D \times \{0\}, \tag{2.4}$$

and the asymptotic condition

$$u, u_{,i} \to 0 \quad \text{as} \quad x_3 \to \infty \quad \text{(uniformly)}. \tag{2.5}$$

We here assume that $K(r)$ is the thermal conductivity.

In all this paper Greek sub-indices are restricted to the values 1 and 2. As usual, summation over repeated sub-indices is assumed and the partial derivative with respect x_i is denoted by "$_{,i}$".

2.1 Estimates

We now characterize the rate of decay for the problem considered below. If we multiply the equation (2.2) by $u_{,3}$ and we integrate along the cylinder, and taking into account the boundary conditions, we obtain the equality

$$\int_{D(z)} K(r)u_{,\alpha}u_{,\alpha}\,da - \int_{D(z)} K(r)u_{,3}^2\,da = E(0) = 0, \tag{2.6}$$

where $D(z) = \{\boldsymbol{x} \in R : x_3 = z\}$ and the last equality is obtained from the asymptotic condition (2.5). From (2.6), we obtain

$$\int_{D(z)} K(r)u_{,\alpha}u_{,\alpha}\,da = \int_{D(z)} K(r)u_{,3}^2\,da. \tag{2.7}$$

If we define the function

$$F(z) = \frac{1}{2}\int_{D(z)} K(r)u^2\,da, \tag{2.8}$$

we see that

$$F''(z) = \int_{D(z)} K(r)u_{,i}u_{,i}\,da = 2\int_{D(z)} K(r)u_{,\alpha}u_{,\alpha}\,da. \tag{2.9}$$

Here, the first equality follows after the use of the equation (2.2), the divergence theorem jointly with the boundary conditions (2.3), (2.4) and the last equality follows from (2.7).

As D is a circular crown of radius a and b, we see that

$$F(z) = \frac{1}{2}\int_0^{2\pi}\int_a^b rK(r)u^2\,drd\theta \tag{2.10}$$

and

$$F''(z) = 2\int_0^{2\pi}\int_a^b rK(r)\left(u_{,r}^2 + \frac{u_{,\theta}^2}{r^2}\right)\,drd\theta. \tag{2.11}$$

Therefore,

$$F''(z) \geq 4\lambda_*^2 F(z), \tag{2.12}$$

where λ_*^2 is the first eigenvalue of the regular Sturm-Liouville problem

$$(rK(r)\phi'(r))' + \lambda^2 rK(r)\phi(r) = 0, \quad \phi(a) = \phi(b) = 0. \tag{2.13}$$

Inequality (2.12) is well known in the study of the Saint-Venant Principle (see [22]) and in view of the asymptotic condition proposed at (2.5), we see that

$$F(z) \leq F(0)\exp(-2\lambda_* z). \tag{2.14}$$

That is, λ_* is the rate of decay for the solutions of our problem.

Remark 2.1 *The analysis proposed here also applies if we allow that $K(r)$ vanishes at a (or b), whenever we consider that the solutions are bounded in the corresponding part of the boundary of D.*

3 A Theoretical Result

In this section we propose a theorem and two corollaries which determine lower bounds for the value of λ_* obtained by (2.14).

Before to propose our main theorem, we recall that the first eigenvalue $\lambda^2_{a,b}$ of the regular Sturm-Liouville problem

$$(r\phi'(r))' + \lambda^2 r\phi(r) = 0, \quad \phi(a) = \phi(b) = 0 \tag{3.1}$$

is given by the smallest solution of the equation

$$J_0(\lambda a)Y_0(\lambda b) = J_0(\lambda b)Y_0(\lambda a), \tag{3.2}$$

being $J_0(x)$ and $Y_0(x)$ the well-known Bessel functions (see e.g. [24]).

We recall that the first eigenvalue of the problem (2.13) can be characterized by means of the minimum value of the Rayleigh quotient (see [24] p.184):

$$\lambda^2_* = \min \frac{\int_a^b rK(r)\left(\phi'(r)\right)^2 dr}{\int_a^b rK(r)\phi^2(r)dr} \tag{3.3}$$

By means of the characterization of the solutions of (3.2), we are in the situation to establish our main theorem. We point out that the argument to show the theorem is very similar to the one proposed in [23]. However, we give the proof to be self-contained. In fact, it is relevant in our approach because it describes the way to obtain the lower bound for the Poincaré constant.

Theorem 3.1 *Let $K^{1/2}(r) \in \mathcal{C}^2(a,b)$ such that $K^{1/2}(r) > 0$ on $[a,b]$. Let us suppose the existence of three constants C_1, C_2 and C_3 such that*

$$\left(K^{1/2}\right)'' \geq -C_1\left(K^{1/2}\right)' - C_2K^{1/2} \quad \text{on } (a,b) \tag{3.4}$$

and

$$\frac{C_1}{2}K^{1/2} + \left(K^{1/2}\right)' \geq -C_3 rK^{1/2} \quad \text{on } (a,b), \tag{3.5}$$

where[1]

$$C_2 + C_3 < \lambda^2_{a,b}. \tag{3.6}$$

Then, there exists a positive constant $k_1 = k_1(|C_1|, C_2, C_3)$ such that

$$\int_a^b rK\phi^2\, dr \leq k_1 \int_a^b rK\left(\phi'\right)^2\, dr \tag{3.7}$$

for all continuous function $\phi(r)$ such that $\phi(a) = \phi(b) = 0$.

Proof. We consider the function $G(r) = K^{1/2}(r)\phi(r)$. We know

$$\int_a^b rG^2\, dr \leq \lambda^{-2}_{a,b} \int_a^b r\left(G'\right)^2 dr. \tag{3.8}$$

[1] $\lambda^2_{a,b}$ is the first eigenvalue of the problem (3.1).

After an integration by parts and recalling the definition of the function $G(r)$, we see that

$$\int_a^b r\left(G'\right)^2 dr = \int_a^b rK\left(\phi'\right)^2\, dr - \int_a^b rK^{1/2}\left(K^{1/2}\right)''\phi^2\, dr - \int_a^b K^{1/2}\left(K^{1/2}\right)'\phi^2\, dr. \tag{3.9}$$

From (3.8) and (3.9), we obtain

$$\begin{aligned}\int_a^b rK\phi^2\, dr \leq \lambda_{a,b}^{-2}\Bigg[\int_a^b rK\left(\phi'\right)^2\, dr &- \int_a^b rK^{1/2}\left(K^{1/2}\right)''\phi^2(r)\, dr \\ &- \int_a^b rK^{1/2}\left(K^{1/2}\right)'\phi^2\, dr\Bigg].\end{aligned} \tag{3.10}$$

In view of (3.4) and the monotonicity of the integration, we have that

$$-\int_a^b rK^{1/2}\left(K^{1/2}\right)''\phi^2\, dr \leq C_1\int_a^b rK^{1/2}\left(K^{1/2}\right)'\phi^2 dr + C_2\int_a^b rK\phi^2 dr. \tag{3.11}$$

But

$$\frac{1}{2}\int_a^b rK'\phi^2 dr = -\int_a^b rK\phi\phi' dr - \frac{1}{2}\int_a^b K\phi^2 dr. \tag{3.12}$$

From (3.10)–(3.12) we obtain

$$\begin{aligned}\int_a^b rK\phi^2 dr \leq \lambda_{a,b}^{-2}&\left[\int_a^b rK\left(\phi'\right)^2 - C_1\int_a^b rK\phi\phi' dr + C_2\int_a^b rK\phi^2 dr\right] \\ &- \lambda_{a,b}^{-2}\int_a^b\left(\frac{C_1}{2}K^{1/2} + \left(K^{1/2}\right)'\right)K^{1/2}\phi^2 dr.\end{aligned} \tag{3.13}$$

If we recall (3.5), the last integral in (3.13) can be estimated by

$$C_3\int_a^b rK\phi^2 dr. \tag{3.14}$$

We then obtain

$$\int_a^b rK\phi^2 dr \leq \lambda_{a,b}^{-2}\left[\int_a^b rK\left(\phi'\right)^2 dr - C_1\int_a^b rK\phi\phi' dr + (C_2 + C_3)\int_a^b rK\phi^2 dr\right]. \tag{3.15}$$

After the use of the arithmetic-geometric mean inequality it follows that

$$\left[1 - \left(C_2 + C_3 + \frac{|C_1|\varepsilon}{2}\right)\lambda_{a,b}^{-2}\right]\int_a^b rK\phi^2 dr \leq \lambda_{a,b}^{-2}\left(1 + \frac{|C_1|}{2\varepsilon}\right)\int_a^b rK\left(\phi'\right)^2 dr, \tag{3.16}$$

where ε is an arbitrary positive constant. We know select ε such that

$$1 - \left(C_2 + C_3 + \frac{|C_1|\varepsilon}{2}\right)\lambda_{a,b}^{-2} > 0. \tag{3.17}$$

We have the estimate

$$\int_a^b rK\phi^2 dr \leq Q(\varepsilon)\int_a^b rK\left(\phi'\right)^2 dr, \tag{3.18}$$

where

$$Q(\varepsilon) = \lambda_{a,b}^{-2} \frac{1 + |C_1|/(2\varepsilon)}{1 - (C_2 + C_3 + |C_1|\varepsilon/2)\, \lambda_{a,b}^{-2}}. \tag{3.19}$$

We can pick ε to minimize $Q(\varepsilon)$ and satisfying (3.17) by choosing

$$\varepsilon^* = -B + \sqrt{B^2 + (1 - CA)/A}, \tag{3.20}$$

where

$$A = \lambda_{a,b}^{-2}, \quad B = |C_1|/2, \quad C = C_2 + C_3, \tag{3.21}$$

we have

$$Q(\varepsilon^*) = \frac{A(1 + B/\varepsilon^*)}{1 - (C + B\varepsilon^*)A}. \tag{3.22}$$

The theorem is proved by taking $k_1 = Q(\varepsilon^*)$, obtained at (3.22). $\square$

When $C_3 = 0$, we obtain the following result.

Corollary 3.1 *Let $K^{1/2}(r) \in \mathcal{C}^2(a,b)$ such that $K^{1/2}(r) > 0$ on $[a,b]$. We assume the existence of two constants C_1 and C_2 such that*

$$\left(K^{1/2}\right)'' \geq -C_1 \left(K^{1/2}\right)' - C_2 K^{1/2} \quad \text{on } (a,b) \tag{3.23}$$

and

$$\frac{C_1}{2} K^{1/2} + \left(K^{1/2}\right)' \geq 0 \quad \text{on } (a,b), \tag{3.24}$$

where

$$C_2 < \lambda_{a,b}^2. \tag{3.25}$$

There exists a positive constant $k_1 = k_1(|C_1|, C_2)$ such that

$$\int_a^b rK\phi^2 dr \leq k_1 \int_a^b rK \left(\phi'\right)^2 dr, \tag{3.26}$$

for all continuous function $\phi(r)$ such that $\phi(a) = \phi(b) = 0$.

It is worth noting that in this case k_1 is given by (3.22), but in this case we take $C = C_2$ in (3.21).

When $C_1 = C_2 = 0$, we obtain the following result.

Corollary 3.2 *Let $K^{1/2}(r) \in \mathcal{C}^2(a,b)$ such that $K^{1/2}(r) > 0$ on $[a,b]$. Let us also assume that*

$$\left(K^{1/2}\right)' \geq 0 \quad \text{and} \quad \left(K^{1/2}\right)'' \geq 0 \quad \text{on } (a,b). \tag{3.27}$$

Then,

$$\int_a^b rK\phi^2 dr \leq \lambda_{a,b}^{-2} \int_a^b rK \left(\phi'\right)^2 dr, \tag{3.28}$$

for all continuous function $\phi(r)$ such that $\phi(a) = \phi(b) = 0$.

Remark 3.1 *The theses of Theorem 3.1 and Corollaries 3.1 and 3.2 also hold in case we allow that the function $K(r)$ vanishes at the point a (respectively b), whenever we assume that $\phi(a)$ (respectively $\phi(b)$) is bounded .*

4 Some Applications

In this section we consider several examples of functions $K(r)$ and we calculate a corresponding lower bound for the rate of decay.

Example 4.1 We consider the function

$$K^{1/2}(r) = K_0^{1/2} \exp\left(\frac{m(r-a)^2}{(b-a)^2}\right), \tag{4.1}$$

where m is a dimensionless real constant. We see

$$\left(K^{1/2}\right)' = K_0^{1/2} \frac{2m(r-a)}{(b-a)^2} \exp\left(\frac{m(r-a)^2}{(b-a)^2}\right) \tag{4.2}$$

and

$$\left(K^{1/2}\right)'' = \left(\frac{4m^2(r-a)^2}{(b-a)^4} + \frac{2m}{(b-a)^2}\right) K_0^{1/2} \exp\left(\frac{m(r-a)^2}{(b-a)^2}\right). \tag{4.3}$$

We first assume that m is less that zero. We have

$$\left(K^{1/2}\right)'' \geq \frac{2m}{(b-a)^2} K^{1/2}. \tag{4.4}$$

Condition (3.4) is satisfied for

$$C_1 = 0 \ \text{ and } \ C_2 = -\frac{2m}{(b-a)^2}. \tag{4.5}$$

We also have

$$\left(K^{1/2}\right)' \geq \frac{2mr}{(b-a)^2} K^{1/2}. \tag{4.6}$$

We can take $C_3 = -\dfrac{2m}{(b-a)^2}$ and (3.5) holds.

As we assume that m is negative, we should impose that

$$C_2 + C_3 = -\frac{4m}{(b-a)^2} < \lambda_{a,b}^2. \tag{4.7}$$

We have that

$$A = \lambda_{a,b}^{-2}, \quad B = 0, \quad C = -\frac{4m}{(b-a)^2} \tag{4.8}$$

and

$$Q(\varepsilon^*) = \frac{\lambda_{a,b}^{-2}}{1 + \dfrac{4m\lambda_{a,b}^{-2}}{(b-a)^2}}. \tag{4.9}$$

We see that the lower bound for the decay is

$$k \equiv \sqrt{\lambda_{a,b}^2 + \frac{4m}{(b-a)^2}}. \tag{4.10}$$

We now consider the case when m is positive. We can use Corollary 3.1 with $C_1 = 0$ and $C_2 = \dfrac{-2m}{(b-a)^2}$. We then obtain

$$k \equiv \sqrt{\lambda_{a,b}^2 + \frac{2m}{(b-a)^2}}. \tag{4.11}$$

In Table 1 we show several values obtained for the calculation of the lower bound for the rate of decay corresponding to Example 4.1, with the fixed small radius $a = 1$ and for several values of the radius b. The Table 1 presents the first eigenvalue $\lambda_{a,b}^2$ of the Sturm-Liouville problem (3.1); the admissible values for the dimensionless parameter m and, in the last two columns, the lower bounds of the rate of decay λ_*, according to the sign of m. Analogously in Table 2 for the fixed radius $a = 2$.

$a = 1$				
b	$\lambda_{1,b}^2$	Condition on $m < 0$	Lower bound if $m < 0$	Lower bound if $m \geq 0$
1.1	31.4123...	$m \geq -0.07853...$	$\sqrt{\lambda_{1,1.1}^2 + 400m}$	$\sqrt{\lambda_{1,1.1}^2 + 200m}$
1.5	6.27024...	$m \geq -0.39189...$	$\sqrt{\lambda_{1,1.5}^2 + 16m}$	$\sqrt{\lambda_{1,1.5}^2 + 8m}$
2	3.12303...	$m \geq -0.78076...$	$\sqrt{\lambda_{1,2}^2 + 4m}$	$\sqrt{\lambda_{1,2}^2 + 2m}$
4	1.02442...	$m \geq -2.30495...$	$\sqrt{\lambda_{1,4}^2 + 4m/9}$	$\sqrt{\lambda_{1,4}^2 + 2m/9}$
20	0.15322...	$m \geq -13.8281...$	$\sqrt{\lambda_{1,20}^2 + 4m/19^2}$	$\sqrt{\lambda_{1,20}^2 + 2m/19^2}$

Table 1: Example 4.1 with $a = 1$

In Figure 1 we have represented the dependence of the lower bound for the decay with respect to the parameters. We have fixed the small radius of the cylinder $a = 1$. This picture corresponds to some values such that $b > a$.

Example 4.2 We consider the function

$$K^{1/2}(r) = K_{00}^{1/2} \exp\left(\frac{m(r-a)}{b-a}\right) + K_{01}^{1/2} r \exp\left(\frac{m(r-a)}{b-a}\right), \tag{4.12}$$

where $K_{00}^{1/2}$, $K_{01}^{1/2}$ are two non-negative constants and m is a dimensionless parameter. We have that

$$\left(K^{1/2}\right)' = \frac{m}{b-a}\left[K_{00}^{1/2} + K_{01}^{1/2}\left(r + \frac{b-a}{m}\right)\right] \exp\left(\frac{m(r-a)}{b-a}\right) \tag{4.13}$$

$a = 2$				
b	$\lambda^2_{2,b}$	Condition on $m < 0$	Lower bound if $m < 0$	Lower bound if $m \geq 0$
2.1	31.415...	$m \geq -0.07854...$	$\sqrt{\lambda^2_{2,2.1} + 400m}$	$\sqrt{\lambda^2_{2,2.1} + 200m}$
2.2	15.7062...	$m \geq -0.157062...$	$\sqrt{\lambda^2_{2,2.2} + 100m}$	$\sqrt{\lambda^2_{2,2.2} + 50m}$
3	3.13512...	$m \geq -0.783779...$	$\sqrt{\lambda^2_{2,3} + 4m}$	$\sqrt{\lambda^2_{2,3} + 2m}$
4	1.56152...	$m \geq -1.56152...$	$\sqrt{\lambda^2_{2,4} + m}$	$\sqrt{\lambda^2_{2,4} + m/2}$
10	0.381596...	$m \geq -6.10553...$	$\sqrt{\lambda^2_{2,10} + m/32}$	$\sqrt{\lambda^2_{2,10} + m/16}$

Table 2: Example 4.1 with $a = 2$

and

$$\left(K^{1/2}\right)'' = \left(\frac{m}{b-a}\right)^2 \left[K_{00}^{1/2} + K_{01}^{1/2}\left(r + 2\,\frac{b-a}{m}\right)\right] \exp\left(\frac{m(r-a)}{b-a}\right). \tag{4.14}$$

So,

$$\left(K^{1/2}\right)'' = \frac{2m}{b-a}\left(K^{1/2}\right)' - \left(\frac{m}{b-a}\right)^2 K^{1/2}. \tag{4.15}$$

The assumptions of Corollary 3.1 are satisfied by taking

$$C_1 = -\frac{2m}{b-a}, \quad C_2 = \left(\frac{m}{b-a}\right)^2. \tag{4.16}$$

At the same time

$$-\frac{m}{b-a}K^{1/2} + \left(K^{1/2}\right)' \geq 0. \tag{4.17}$$

Condition (3.25) is satisfied whenever

$$m^2 \leq (b-a)^2\lambda^2_{a,b}. \tag{4.18}$$

We have that

$$A = \lambda^{-2}_{a,b}, \quad B = \frac{|m|}{b-a}, \quad C = \left(\frac{m}{b-a}\right)^2. \tag{4.19}$$

We see that

$$\varepsilon^* = \lambda_{a,b} - \frac{|m|}{b-a} \tag{4.20}$$

and we obtain

$$Q(\varepsilon^*) = \frac{\lambda_{a,b}}{\left(\lambda^2_{a,b} - C - B\varepsilon^*\right)\varepsilon^*}. \tag{4.21}$$

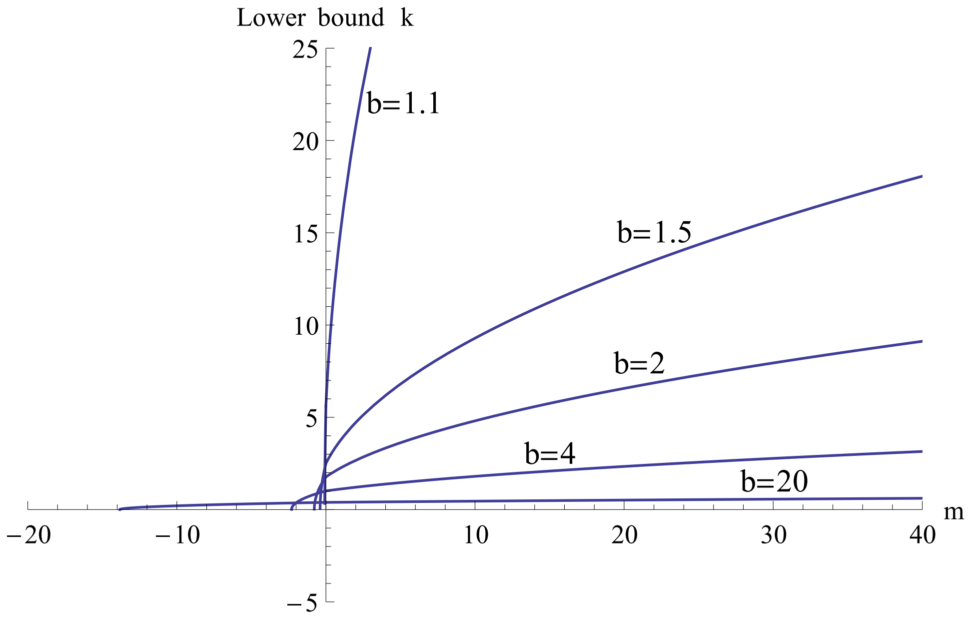

Figure 1: *Lower bound for the decay for Example 4.1 with $a = 1$ and some b.*

Therefore, the lower bound for the decay is

$$k \equiv \sqrt{\lambda_{a,b}^{-1}\left(\lambda_{a,b}^2 - C - B\varepsilon^*\right)\varepsilon^*} = \sqrt{\left(\lambda_{a,b} - \frac{|m|}{b-a}\right)^2} = \left|\lambda_{a,b} - \frac{|m|}{b-a}\right|. \tag{4.22}$$

Since we have that $\lambda_{a,b}$ and $\frac{|m|}{b-a}$ are positive, from condition (4.18) we have $\lambda_{a,b} > \frac{|m|}{b-a}$ and, hence, the lower bound becomes

$$k \equiv \lambda_{a,b} - \frac{|m|}{b-a}. \tag{4.23}$$

The domain of admissible values for m is given by (4.18).

Table 3 contains information concerning to Example 4.2, with the fixed small radius $a = 1$ and $a = 2$. For several values of the radius b, we obtain $\lambda_{a,b}$, the square root of the first eigenvalue of the Sturm-Liouville problem (3.1); the admissible values for the dimensionless parameter m and, in the fourth and the last columns, the lower bounds of the rate of decay λ_*.

Figure 2 illustrates the dependence of the lower bound for the decay with respect to the parameters. As above, we have fixed the small radius of the cylinder $a = 1$ and $a = 2$ (respectively) and we consider several radius $b > a$.

If $K_{01} \equiv 0$, we have a lower bound for the rate of decay when

$$K^{1/2}(r) = K_{00}^{1/2} \exp\left(\frac{m(r-a)}{b-a}\right)$$

which is given at (4.23).

It is worth noting that the function

$$K_{00} \exp\left(\frac{m(r-a)}{b-a}\right) + K_{01}(r-a)\exp\left(\frac{m(r-a)}{b-a}\right), \tag{4.24}$$

$a=1$				$a=2$			
b	$\lambda_{1,b}$	$\lvert m\rvert \leq$	k	b	$\lambda_{2,b}$	$\lvert m\rvert \leq$	k
1.1	5.60467...	0.560467...	$\lambda_{1,b}-10\lvert m\rvert$	2.1	5.60491...	0.560491...	$\lambda_{2,b}-10\lvert m\rvert$
1.5	2.50404...	1.25202...	$\lambda_{1,b}-2\lvert m\rvert$	2.5	2.50584...	1.25292...	$\lambda_{2,b}-2\lvert m\rvert$
2	1.76721...	1.76721...	$\lambda_{1,b}-\lvert m\rvert$	3	1.77063...	1.77063...	$\lambda_{2,b}-\lvert m\rvert$
4	1.01214...	3.03641...	$\lambda_{1,b}-\dfrac{\lvert m\rvert}{3}$	4	1.24961...	2.49921...	$\lambda_{2,b}-\dfrac{\lvert m\rvert}{2}$
20	0.391434...	7.43724...	$\lambda_{1,b}-\dfrac{\lvert m\rvert}{19}$	20	0.407059...	7.32706...	$\lambda_{2,b}-\dfrac{\lvert m\rvert}{18}$

Table 3: Example 4.2 with $a=1$ and $a=2$

can be written as

$$K_{00}^{*}\exp\left(\frac{m(r-a)}{b-a}\right)+K_{01}r\exp\left(\frac{m(r-a)}{b-a}\right), \tag{4.25}$$

where $K_{00}^{*}=K_{00}-aK_{01}$. Therefore, the results of this example also apply to the family (4.24) whenever $K_{01}\geq 0$ and $K_{00}\geq aK_{01}$.

When $m\to 0$, the family (4.12) becomes

$$K^{1/2}(r)=K_{00}^{1/2}+K_{01}^{1/2}r. \tag{4.26}$$

In this situation, we see that the lower bound for de decay is

$$k\equiv\lambda_{a,b}. \tag{4.27}$$

Example 4.3 We consider the function

$$K^{1/2}(r)=K_{01}^{1/2}\exp\left(\frac{m_1(r-a)}{b-a}\right)+K_{02}^{1/2}\exp\left(\frac{m_2(r-a)}{b-a}\right), \tag{4.28}$$

where $K_{01}^{1/2}$, $K_{02}^{1/2}$ are two non-negative constants and m_1, m_2 are dimensionless constants. We have that

$$\left(K^{1/2}\right)''=\frac{m_1+m_2}{b-a}\left(K^{1/2}\right)'-\frac{m_1m_2}{(b-a)^2}K^{1/2}. \tag{4.29}$$

We can take

$$C_1=-\frac{m_1+m_2}{b-a},\quad C_2=\frac{m_1m_2}{(b-a)^2}. \tag{4.30}$$

We also obtain that

$$\begin{aligned}&\frac{1}{2}C_1K^{1/2}+\left(K^{1/2}\right)'\\&=\frac{m_2-m_1}{2(b-a)}\left[K_{02}^{1/2}\exp\left(\frac{m_2(r-a)}{b-a}\right)-K_{01}^{1/2}\exp\left(\frac{m_1(r-a)}{b-a}\right)\right].\end{aligned} \tag{4.31}$$

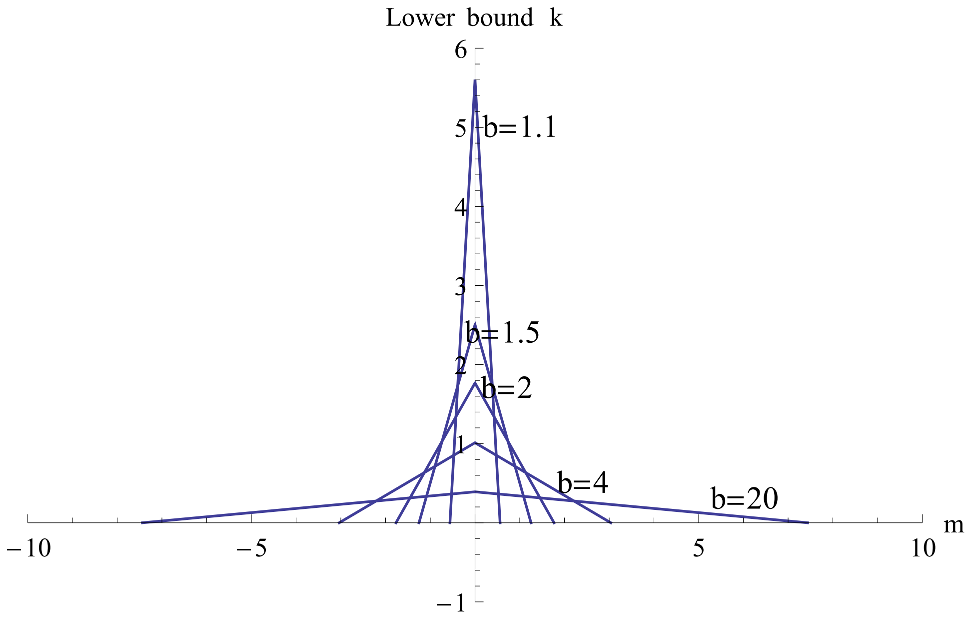

Figure 2: *Lower bound for the decay for Example 4.2 with $a = 1$ and some b.*

So, the right hand side of (4.31) is non-negative whenever

$$(m_2 - m_1)\left(K_{02}^{1/2} - K_{01}^{1/2}\right) \geq 0. \tag{4.32}$$

On the other side, the condition (3.25) reads

$$m_1 m_2 < \lambda_{a,b}^2 (b-a)^2. \tag{4.33}$$

In this case we have

$$A = \lambda_{a,b}^{-2}, \quad B = \frac{|m_1 + m_2|}{2(b-a)}, \quad C = \frac{m_1 m_2}{(b-a)^2} \tag{4.34}$$

and

$$\varepsilon^* = -\frac{|m_1 + m_2|}{2(b-a)} + \sqrt{\lambda_{a,b}^2 + \frac{(m_1 - m_2)^2}{4(b-a)^2}}. \tag{4.35}$$

Therefore,

$$Q(\varepsilon^*) = \frac{\varepsilon^* + B}{\varepsilon^*(\lambda_{a,b}^2 - C - B\varepsilon^*)} \tag{4.36}$$

and the lower bound for the rate of decay is

$$k \equiv \sqrt{\left(\sqrt{\lambda_{a,b}^2 + \frac{(m_1 - m_2)^2}{4(b-a)^2}} - \frac{|m_1 + m_2|}{2(b-a)}\right)^2}. \tag{4.37}$$

From (4.33) it follows that

$$\sqrt{\lambda_{a,b}^2 + \frac{(m_1 - m_2)^2}{4(b-a)^2}} \geq \frac{|m_1 + m_2|}{2(b-a)}. \tag{4.38}$$

Therefore, the lower bound becomes

$$k \equiv \sqrt{\lambda_{a,b}^2 + \frac{(m_1 - m_2)^2}{4(b-a)^2}} - \frac{|m_1 + m_2|}{2(b-a)}. \tag{4.39}$$

$$\tag{4.40}$$

A relevant subcase can be considered when $m_1 = -m_2 = m > 0$. We also have to assume that $K_{01}^{1/2} \geq K_{02}^{1/2}$ in order to accomplish (4.32). Since (4.39), we get directly the lower bound for the rate of decay

$$k \equiv \sqrt{\lambda_{a,b}^2 + \frac{m^2}{(b-a)^2}}. \tag{4.41}$$

When $K_{02} \equiv 0$, we obtain another lower bound for the rate of decay for

$$K^{1/2}(r) = K_{01}^{1/2} \exp\left(\frac{m(r-a)}{b-a}\right)$$

which is given by (4.41). We note that this is faster that the one obtained by means of the Example 4.2.

Furthermore, as a variant of Example 4.3, we shall consider functions of the type

$$K^{1/2}(r) = K_{01}^{1/2} \cosh\left(\frac{m(r-a)}{b-a}\right) + K_{02}^{1/2} \sinh\left(\frac{m(r-a)}{b-a}\right), \tag{4.42}$$

where $K_{01}^{1/2}$ and $K_{02}^{1/2}$ are real constants. Therefore, we can write them as

$$K^{1/2}(r) = \frac{1}{2}\left(K_{01}^{1/2} + K_{02}^{1/2}\right) \exp\left(\frac{m(r-a)}{b-a}\right) + \frac{1}{2}\left(K_{01}^{1/2} - K_{02}^{1/2}\right) \exp\left(-\frac{m(r-a)}{b-a}\right). \tag{4.43}$$

So, if we assume that $K_{01}^{1/2} \geq \left|K_{02}^{1/2}\right|$, we can apply the previous arguments.

$$\tag{4.44}$$

Figure 3 shows the dependence of the lower bound (4.41) for the decay with respect to the parameters for fixed $a = 1$ and several values of b.

Example 4.4 Let us consider the functions

$$K^{1/2}(r) = K_0^{1/2}\left(1 + \frac{m(r-a)}{b-a}\right)^{\beta}, \ \beta \geq 1, \tag{4.45}$$

where $K_0^{1/2}$ is a non-negative constant and m is a dimensionless positive constant. Since $\left(K^{1/2}\right)' \geq 0$ and $\left(K^{1/2}\right)'' \geq 0$, from Corollary 3.2 we get the lower bound for the decay

$$k \equiv \lambda_{a,b}. \tag{4.46}$$

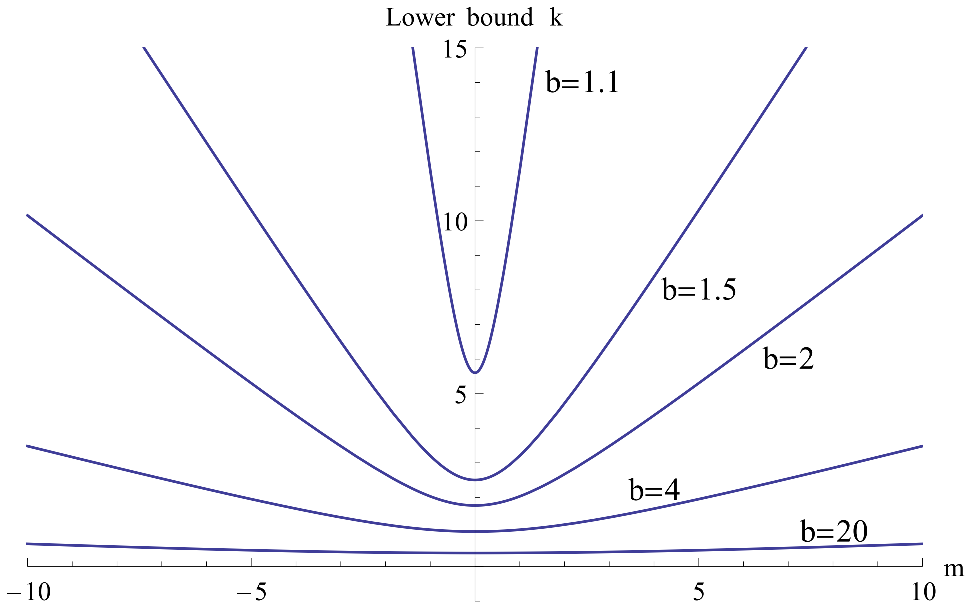

Figure 3: *Lower bound for the decay for Example 4.3 with $a = 1$ and some b.*

Remark 4.1 *Suppose that $K^{1/2}(r)$ is a function such that*

$$0 \leq K_m^{1/2}\,\overline{K}^{1/2}(r) \leq K^{1/2}(r) \leq K_M^{1/2}\,\overline{K}^{1/2}(r), \tag{4.47}$$

where $K_m^{1/2}$, $K_M^{1/2}$ are positive and $\overline{K}^{1/2}(r)$ satisfies the inequality

$$\int_0^a r\,\overline{K}\phi^2 dr \leq k^* \int_0^a r\,\overline{K}\,(\phi')^2\,dr, \tag{4.48}$$

for all continuous function $\phi(r)$ such that $\phi(a) = \phi(b) = 0$. Therefore, we can obtain a similar inequality for $K(r)$. In fact, we have

$$\int_0^a rK\phi^2 dr \leq K_M \int_0^a r\,\overline{K}\phi^2 dr \leq \frac{K_M}{K_m} k^* \int_0^a rK\,(\phi')^2\,dr. \tag{4.49}$$

In the following example we can not directly apply the theorem or the corollaries, but the previous remark.

Example 4.5 Let us consider the family of functions

$$K^{1/2}(r) = K_0^{1/2}\left(1 + \frac{m(r-a)}{b-a}\right)^{\beta},\ 0 < \beta \leq 1, \tag{4.50}$$

where $K_0^{1/2}$ is a non-negative constant and m is a dimensionless positive constant. Since $\beta - 1 \leq 0$, we note that

$$(1+m)^{\beta-1} \leq \left(1 + \frac{m(r-a)}{b-a}\right)^{\beta-1} \leq 1. \tag{4.51}$$

So, we obtain

$$K_0^{1/2}(1+m)^{\beta-1}\left(1+\frac{m(r-a)}{b-a}\right) \le K_0^{1/2}\left(1+\frac{m(r-a)}{b-a}\right)^{\beta} \le K_0^{1/2}\left(1+\frac{m(r-a)}{b-a}\right) \quad (4.52)$$

and (4.47) holds for

$$\overline{K}^{1/2}(r) = 1+\frac{m(r-a)}{b-a}, \quad K_m^{1/2} = K_0^{1/2}(1+m)^{\beta-1}, \quad K_M^{1/2} = K_0^{1/2}. \quad (4.53)$$

Thus, from Remark 4.1 and Example 4.4 (when $\beta = 1$), a lower bound of the decay rate is

$$\lambda_{a,b}(1+m)^{\beta-1}. \quad (4.54)$$

On the other hand, when $\beta > 0$, we have that

$$K_0^{1/2} < K_0^{1/2}\left(1+\frac{m(r-a)}{b-a}\right)^{\beta} < K_0^{1/2}(1+m)^{\beta}. \quad (4.55)$$

Now, (4.47) holds for

$$\overline{K}^{1/2}(r) = 1, \quad K_m^{1/2} = K_0^{1/2}, \quad K_M^{1/2} = K_0^{1/2}(1+m)^{\beta}. \quad (4.56)$$

Hence, from Remark 4.1, we obtain another lower bound of the decay rate:

$$\lambda_{a,b}(1+m)^{-\beta}. \quad (4.57)$$

Clearly, $(1+m)^{\beta-1} = (1+m)^{-\beta}$ when $\beta = 1/2$. By comparing both lower bounds, (4.54) and (4.57), for the values of $\beta \in (0,1]$, we obtain

$$k \equiv \begin{cases} \lambda_{a,b}(1+m)^{-\beta}, & \text{if } 0 < \beta \le 1/2, \\ \lambda_{a,b}(1+m)^{\beta-1}, & \text{if } 1/2 \le \beta \le 1. \end{cases} \quad (4.58)$$

Trivially, when $\beta = 1$, (4.46) and (4.58) agree.

Figure 4 shows the dependence of the lower bound k (4.58) with respect to the parameters for fixed $a = 1$, when $\beta = 1/2$ and several values of b.

Below we propose the last example of this section. We study the family of functions of Examples 4.4 and 4.5 when the parameter β is non-positive.

Example 4.6 Let us consider

$$K^{1/2}(r) = K_0^{1/2}\left(1+\frac{m(r-a)}{b-a}\right)^{\beta}, \ \beta \le 0, \quad (4.59)$$

where $K_0^{1/2}$ is a non-negative constant and m is a dimensionless positive constant.

Since $\beta \le 0$, from

$$1 < 1+\frac{m(r-a)}{b-a} < 1+m, \quad (4.60)$$

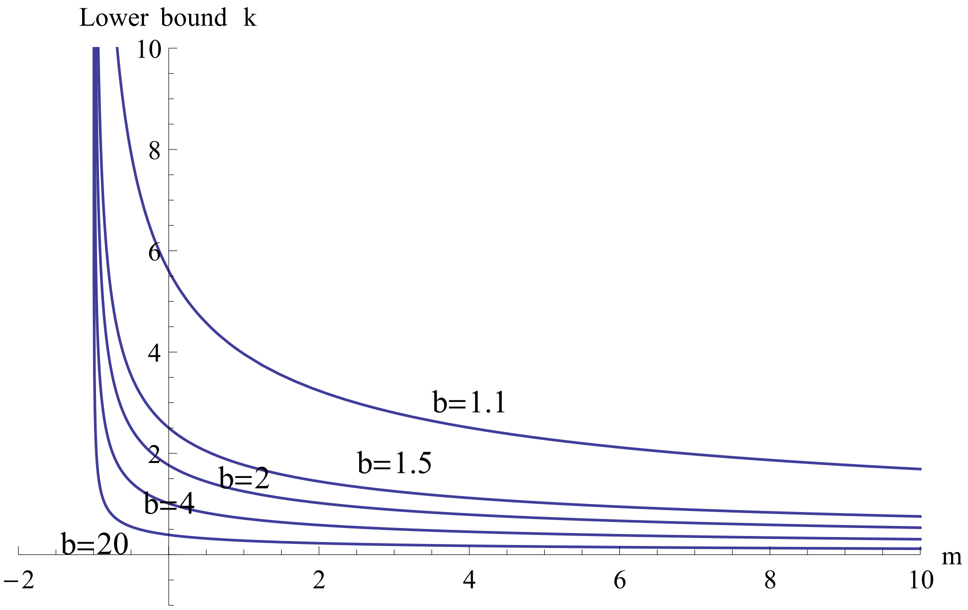

Figure 4: *Lower bound for the decay for Example 4.5 with $a = 1$, $\beta = 1/2$ and some b.*

we get

$$K_0^{1/2}(1+m)^\beta \leq K_0^{1/2}\left(1+\frac{m(r-a)}{b-a}\right)^\beta \leq K_0^{1/2} \tag{4.61}$$

and (4.47) holds by taking

$$\overline{K}^{1/2}(r) = 1, \quad K_m^{1/2} = K_0^{1/2}(1+m)^\beta, \quad K_M^{1/2} = K_0^{1/2}. \tag{4.62}$$

From Remark 4.1, the lower bound of the decay is

$$k \equiv \lambda_{a,b}(1+m)^\beta. \tag{4.63}$$

5 Inhomogeneity also in axial direction

When we assume that the thermal conductivity also depends on the axial variable, the problem to be studied is determined by the equation

$$(K(r,x_3)u_{,\alpha})_{,\alpha} + (K(r,x_3)u_{,3})_{,3} = 0 \text{ in } R \tag{5.1}$$

with the boundary conditions (2.3)–(2.4) and the asymptotic condition (2.5). In this case we cannot apply the previous arguments, but it is possible to adapt them whenever we assume:

(I) $\dfrac{\partial^2\left(K^{1/2}\right)}{\partial r^2} \geq -C_1(x_3)\dfrac{\partial\left(K^{1/2}\right)}{\partial r} - C_2(x_3)K^{1/2}$,

(II) $\dfrac{C_1(x_3)}{2}K^{1/2} + \dfrac{\partial\left(K^{1/2}\right)}{\partial r} \geq -rC_3(x_3)K^{1/2}$, where $C_1(x_3)$, $C_2(x_3)$ and $C_3(x_3)$ are three functions such that

$$C_2(x_3) + C_3(x_3) < \lambda_{a,b}^2, \quad \text{for every } \ x_3 \geq 0. \tag{5.2}$$

It is worth noting that we can obtain the Poincaré type inequality

$$\int_a^b rK(r,x_3)\phi^2 dr \leq k_1(|C_1(x_3)|, C_2(x_3), C_3(x_3)) \int_a^b rK(r,x_3)\phi_{,r}^2 dr \tag{5.3}$$

for every function ϕ vanishing on the end points[2].

The function k_1 can be obtained by means of

$$k_1(x_3) = \frac{A\,(1 + B(x_3)/\varepsilon^*(x_3))}{1 - (C(x_3) + B(x_3)\varepsilon^*(x_3))\,A}, \tag{5.4}$$

where

$$A = \lambda_{a,b}^{-2}, \quad B(x_3) = \frac{|C_1(x_3)|}{2}, \quad C(x_3) = C_2(x_3) + C_3(x_3) \tag{5.5}$$

and

$$\varepsilon^*(x_3) = -B + \sqrt{B^2 + (1 - AC)/A}. \tag{5.6}$$

In this situation, we can obtain spatial decay estimate for the problem determined by the equation (5.1) with the boundary conditions (2.3)–(2.4) together with the asymptotic condition (2.5).

Now, we define the function

$$H(z) = -\int_{D(z)} Kuu_{,3}da = -\int_0^{2\pi}\int_a^b rKuu_{,3}dr\,d\theta. \tag{5.7}$$

We know that

$$H'(z) = \int_{D(z)} K\left(u_{,\alpha}u_{,\alpha} + u_{,3}^2\right)da = \int_0^{2\pi}\int_a^b rK\left(u_{,r}^2 + \frac{u_{,\theta}^2}{r^2} + u_{,3}^2\right)dr\,d\theta. \tag{5.8}$$

As

$$|H(z)| \leq \left(\int_0^{2\pi}\int_a^b rKu^2 dr\,d\theta\right)^{1/2}\left(\int_0^{2\pi}\int_a^b rKu_{,3}^2 dr\,d\theta\right)^{1/2}, \tag{5.9}$$

we can conclude that

$$|H(z)| \leq \frac{1}{2}k_1^{1/2}(z)H'(z). \tag{5.10}$$

From this inequality and the asymptotic condition, we conclude that the function (see [7–9])

$$E(z) = \int_z^\infty K(r,z)|\nabla u|^2 dv \tag{5.11}$$

satisfies the estimate

$$E(z) \leq E(0)\exp\left[-2\int_0^z k_1^{-1/2}(\xi)d\xi\right]. \tag{5.12}$$

[2]Note that this condition can be weaken to assume that the function is bounded whenever we assume that $K(r,z)$ vanishes at the boundary.

If we denote by

$$\bar{k}(z) = \int_0^z k_1^{-1/2}(\xi) d\xi, \tag{5.13}$$

we have seen that a lower bound for the spatial decay is is controlled by the inverse of

$$\exp\left(\bar{k}(z)\right). \tag{5.14}$$

Example 5.1 An elementary example corresponds to the function

$$K^{1/2}(r,z) = K_{00}^{1/2}(z) + rK_{01}^{1/2}(z), \tag{5.15}$$

where $K_{00}^{1/2}$ and $K_{01}^{1/2}$ are non-negative functions. In this situation we see that

$$\frac{\partial\left(K^{1/2}\right)}{\partial r} = K_{01}^{1/2}(z), \quad \frac{\partial^2\left(K^{1/2}\right)}{\partial r^2} = 0. \tag{5.16}$$

By taking $C_i(z) = 0$, for $i = 1, 2, 3$, conditions (I) and (II) hold. We have that $k_1 = \lambda_{a,b}^{-2}$ and

$$\exp\left(\bar{k}(z)\right) = \exp\left(\lambda_{a,b}\, z\right). \tag{5.17}$$

It is worth noting that (5.15) can also be written as

$$K_{00}^{*1/2}(z) + (r-a)K_{01}^{1/2}(z), \tag{5.18}$$

where $K_{00}^{*1/2}(z) = K_{00}^{1/2}(z) + aK_{01}^{1/2}(z)$.

Example 5.2 A second example could be

$$K^{1/2}(r,z) = K_{01}^{1/2}(z)\cosh\left(\frac{m(r-a)}{b-a}\right) + K_{02}^{1/2}(z)\sinh\left(\frac{m(r-a)}{b-a}\right), \tag{5.19}$$

where $m > 0$ and $K_{01}^{1/2}(z) \geq \left|K_{02}^{1/2}(z)\right|$. In this case, from the ideas developed in Example 4.3 by considering functions of the type (4.42), we can obtain the lower bound for the decay

$$\exp\left(\bar{k}(z)\right) = \exp\left(\sqrt{\lambda_{a,b}^2 + \frac{m^2}{(b-a)^2}}\; z\right). \tag{5.20}$$

Example 5.3 The third example we consider is given by the function

$$K^{1/2}(r,z) = K_{01}^{1/2}\cosh\left[m\left(\frac{z}{b-a}\right)^{1/2}\frac{r-a}{b-a}\right] + K_{02}^{1/2}\sinh\left[m\left(\frac{z}{b-a}\right)^{1/2}\frac{r-a}{b-a}\right]. \tag{5.21}$$

There we assume that m is positive and $K_{01}^{1/2} \geq \left|K_{02}^{1/2}\right|$. In this situation,

$$\frac{\partial\left(K^{1/2}\right)}{\partial r} \geq 0 \quad \text{and} \quad \frac{\partial^2\left(K^{1/2}\right)}{\partial r^2} = \left(\frac{m}{b-a}\right)^2 \frac{z}{b-a} K^{1/2}. \tag{5.22}$$

We can take $C_1(z) = C_3(z) = 0$ and $C_2(z) = -\left(\frac{m}{b-a}\right)^2 \frac{z}{b-a}$. Therefore, we obtain

$$k_1(z) = \left[\lambda_{a,b}^2 + \frac{m^2 z}{(b-a)^3}\right]^{-1}. \tag{5.23}$$

So, we get

$$\begin{aligned} \bar{k}(z) =& \lambda_{a,b} \int_0^z \left(1 + \frac{m^2 \xi}{(b-a)^3 \lambda_{a,b}^2}\right)^{1/2} d\xi \\ =& \frac{2}{3} \frac{\lambda_{a,b}^3 (b-a)^3}{m^2} \left[\left(1 + \frac{m^2 z}{(b-a)^3 \lambda_{a,b}^2}\right)^{3/2} - 1\right]. \end{aligned} \tag{5.24}$$

We can approximate the lower bound for the decay as follows

$$\exp\left(\bar{k}(z)\right) \sim \exp\left[\frac{2}{3} m \left(\frac{z}{b-a}\right)^{3/2}\right]. \tag{5.25}$$

This bound is faster than an exponential of a linear function.

6 Increasing cross-section

We can also consider the case when the cross-section (circular crown) is an increasing function in the axial direction. That is, we consider the problem determined in the region

$$\{(r, x_3) | x_3 \geq 0, a(x_3) \leq r \leq b(x_3)\}, \tag{6.1}$$

where $a(x_3)$ and $b(x_3)$ are strictly positive functions.

We assume the boundary conditions

$$u(a(x_3), x_3) = u(b(x_3), x_3) = 0, \tag{6.2}$$

together with the asymptotic condition (2.5). If we define the function

$$H(z) = -\int_{D(z)} K u u_{,3} da = -\int_0^{2\pi} \int_{a(z)}^{b(z)} r K u u_{,3} dr dz, \tag{6.3}$$

we have that

$$H'(z) = \int_{D(z)} K \left|\nabla u\right|^2 da = \int_0^{2\pi} \int_{a(z)}^{b(z)} r K \left(u_{,r}^2 + \frac{u_{,\theta}^2}{r^2} + u_{,3}^2\right) dr dz. \tag{6.4}$$

And so we can obtain

$$|H(z)| \leq \frac{1}{2} k_1^{1/2}(z) H'(z), \tag{6.5}$$

where

$$k_1 = k_1(|C_1(z)|, C_2(z), C_3(z), a(z), b(z)). \tag{6.6}$$

In this case

$$A(z) = \lambda^{-2}_{a(z),b(z)},\; B(z) = \frac{C_1(z)}{2},\; C(z) = C_2(z) + C_3(z). \tag{6.7}$$

Whenever we assume

$$C_2(z) + C_3(z) \leq \lambda^2_{a(z),b(z)}, \tag{6.8}$$

we can obtain a decay estimate. In case that $b(z) - a(z)$ increases to infinity, we know that $\lambda^2_{a(z),b(z)}$ tends to zero. Therefore, to avoid cumbersome situations, we will consider the case

$$C_2(z) + C_3(z) \leq 0. \tag{6.9}$$

For the Example 5.2 we see that the spatial decay is controlled by the inverse of

$$\bar{k}(z) = \int_0^z \sqrt{\lambda^2_{a(\xi),b(\xi)} + \frac{m^2}{(b(\xi) - a(\xi))^2}}\, d\xi. \tag{6.10}$$

First, we consider $a = 1$ and $b = 2(1 + \xi)$, where $\xi = 0.1n$, for $n = 0, 1, 2, ..., 100$. For these parameters, we calculate the eigenvalues $\lambda^2_{a(\xi),b(\xi)}$ and we represent them in Figure 5. We want to approximate the function $\bar{k}(z)$ for some values of m in a certain interval. To this end, we consider the interval $[0, 10]$ and the partition given by $z = 0.1n$, for $n = 0, 1, 2, ..., 100$. We approximate the integral $\bar{k}(z)$ by means of the upper sum and the lower sum corresponding to the above partition, for $m = 0, 2, 5$. These sums are given by the graphs of the Figure 6. So, a discretization of the graph of $\bar{k}(z)$ is located between $k_{\inf}$ and $k_{\sup}$.

Second, we take the two variable radiuses $a = 1 + \xi$ and $b = 2(1 + \xi)$, where $\xi = 0.1n$, for $n = 0, 1, 2, ..., 100$. Analogously to the previous case, we obtain the approximation of $\bar{k}(z)$ in Figure 7.

Acknowledgments

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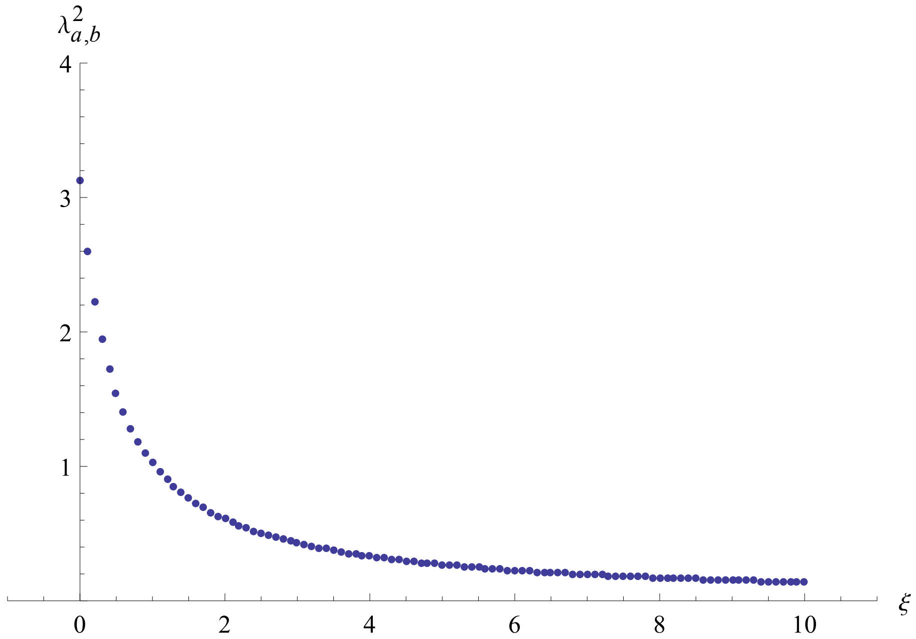

Figure 5: *The eigenvalues* $\lambda^2_{a(\xi),b(\xi)}$ *for* $a = 1$ *and* $b = 2(1+\xi)$.

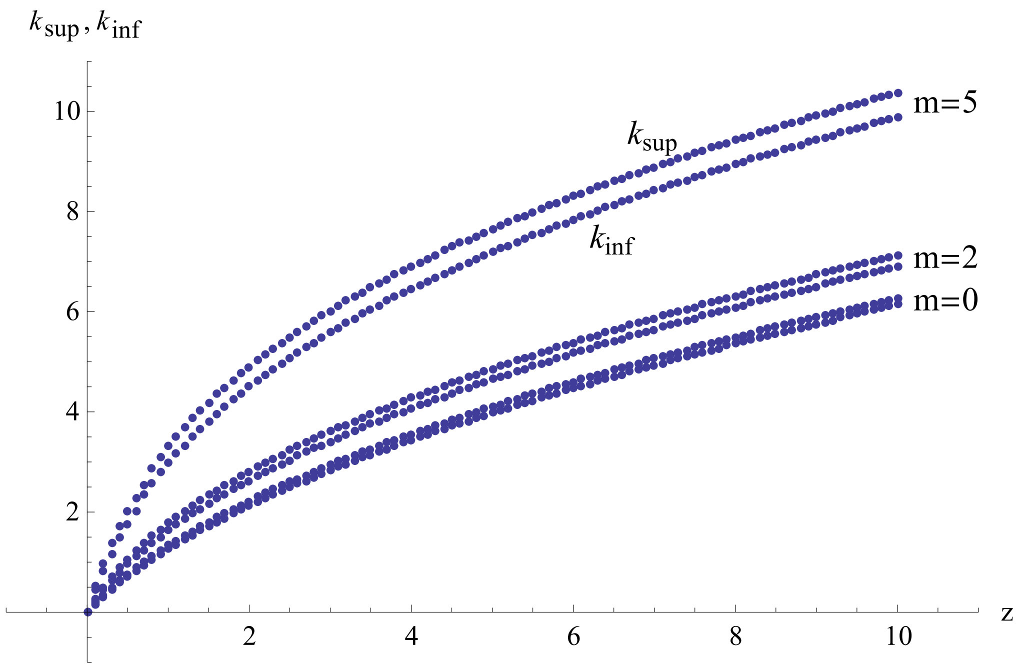

Figure 6: *Upper and lower sums for* $m = 0, 2, 5,$ *when* $a = 1$ *and* $b = 2(1+\xi)$.

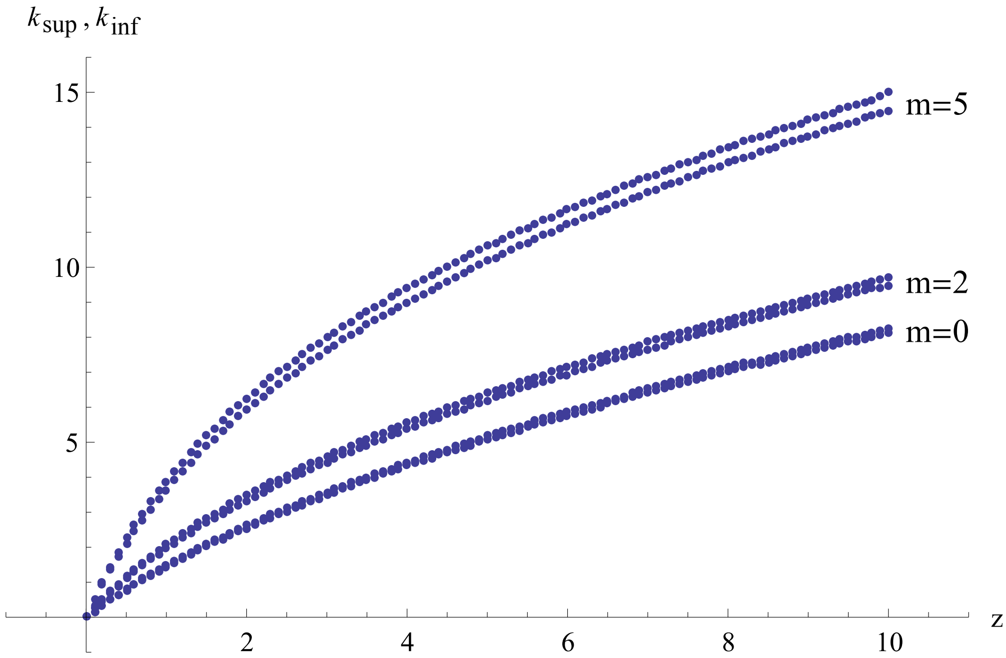

Figure 7: *Upper and lower sums for $m = 0, 2, 5$, when $a = 1 + \xi$ and $b = 2(1 + \xi)$.*

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